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Symbolism, combinations, and visual imagery in the mathematics of Thomas Harriot

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Abstract

The mathematical work of Thomas Harriot (c. 1560–1621) is distinguished by extensive use of symbolism and other forms of visual imagery and by systematic use of combinations. This paper argues that these characteristics of his mathematical writing were already observable in the mid-1580s, in the phonetic alphabet he devised to record the speech of American Indians. The paper presents several little-known examples of Harriot's mathematics, demonstrating his use of symbolism both as a means of expression and as an analytic tool, and assesses Harriot's work in relation to the broader 17th-century trend toward symbolization in mathematics.

Résumé

Le travail mathématique de Thomas Harriot (c. 1560–1621) se remarque par l'usage qu'il fait de symbolismes et d'autres formes d'images visuelles, et par l'utilisation systématique de combinaisons. Cet exposé argumente que ces traits de son écriture mathématique s'observaient déjà à partir de 1585, dans l'alphabet phonétique qu'il avait conçu afin d'enregistrer la parole des Indiens d'Amérique. L'exposé présente plusieurs exemples peu connus des mathématiques de Harriot et montre son utilisation du symbolisme à la fois en tant que moyen d'expression et outil analytique. Finalement il évalue la place de son travail par rapport à la tendance générale vers la symbolisation mathématique au dix-septième siècle.

MSC: 01A45

Introduction

One of the hallmarks of Thomas Harriot's mathematical writing is his extensive use of symbolism and other forms of visual imagery such as lists or tables; another is his systematic use of combinations. This paper explores these themes by examining some of Harriot's mathematical writings from around 1600 onward in arithmetic, geometry, algebra, and the study of combinations and permutations. Most of the examples presented here have not been published or discussed in detail previously and therefore offer fresh insights into Harriot's mathematical thinking and into the range and variety of his achievements. The primary focus of the paper, however, is Harriot's skill in devising and using appropriate symbolism, and I will argue that in this respect some of the most significant characteristics of his mature

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mathematics were already observable in a quite different context in the mid-1580s, in his phonetic alphabet designed to record the speech of American Indians.

Harriot's Algonquin alphabet

When Arthur Barlowe and Philip Amadas returned to England in September 1584 from the first English expedition to the coast of what is now North Carolina, they brought with them two Algonquin Indians, Manteo and Wanchese. By December of that year Harriot was able to understand some of their speech, and in 1585 he devised a phonetic alphabet for writing down its sounds. Some 16 years earlier, John Hart had already experimented with writing English phonetically, using the ordinary alphabet with some additional signs [Hart, 1569, 43v–67]. We do not know whether or not Harriot knew of Hart's work, but he went further by abandoning the English alphabet altogether and creating entirely new symbols.

Harriot's alphabet survives in full on two separate sheets, each of which carries the same symbols, and the same set of words written in them. One of the sheets, dated 1585, also explains the symbols by suggesting English words with similar sounds (see Fig. 1). This dated sheet came into the possession of Richard Busby, headmaster of Westminster School, London, in the later part of the 17th century and lay forgotten or unrecognized in the school library until the late 1980s. The second sheet, which is undated, is among Harriot's manuscripts in the British Library [BL Add MS 6782, f. 337]. Occasional words or phrases in the same script are to be found scattered elsewhere in Harriot's manuscripts [see Seaton, 1956; Shirley, 1983, 108–112; Wallace, 1985].

An ordinary alphabet in any language consists of arbitrary characters, usually between twenty and thirty in number, each associated with one or more sounds.[1] A phonetic alphabet works on the same principle, with just one sound for each symbol, but as in an ordinary alphabet the symbols themselves may be arbitrarily chosen. Alec Wallace in the late 1970s recognized that Harriot's symbols were phonetic and observed that Harriot arranged the sounds in twos or threes, with pairings, for instance, between voiced and unvoiced consonants, such as "p" and "b", or "t" and "d" [Wallace, 1985]. John Shirley, in his 1983 biography of Harriot, drew on Wallace's findings and correlated Harriot's symbols with their modern phonetic counterparts but did not explain or investigate the associations any further [Shirley, 1983, 110–112]. The rediscovery of the Westminster manuscript a few years later aroused new interest in the alphabet and in 1992 Vivian Salmon went some way towards explaining Harriot's system, but his analysis was incomplete and in certain respects possibly unhelpful, and I shall return to it later [Salmon, 1992].

Like all previous commentators I will avoid any analysis of the vowels, except to say that Harriot's 10 vowel sounds were arranged in 5 pairs, with related symbols for each pair. However, the pronunciation of vowels varies significantly with time and geographical location, and only an expert can comment on their sounds in Elizabethan London English. The consonants, on the other hand, are more stable, and here we can follow Harriot's thinking more clearly. First we observe that each of his consonant symbols may be separated into three parts: a central portion, an optional upper loop, and an optional lower loop. At least one loop is used in every consonant (the vowels do not have them). The symbols that have both an upper and lower loop (perhaps suggesting some kind of restriction) represent the sounds now called *stops*: "b", "p", "d", "t", "g", "k". The central elements of those symbols are shown in the first row of Fig. 2, together with Harriot's descriptions of the corresponding sound in brackets alongside each. Two patterns emerge immediately. First, we see that pairs of voiced and unvoiced sounds are written using pairs of signs, each of which is a simple variation of its partner. Second, the left-to-right ordering is Harriot's, and it is clear that the symbols to the left represent the sounds made furthest forward in the mouth while those to the right represent the sounds furthest back. These patterns remain, with only small variations, in all the other rows of the table.

The second and third rows contain sounds to which Harriot assigned only an upper loop. Those in the second row are now classed as *fricatives*: "v", "f", hard "th" (as in "the"), and soft "th" (as in "thorn"), together with the additional sounds Harriot described as "gh" and "ch". These last two had no exact equivalents in English. The first, presumably a guttural sound, he described only as occurring "in some barbarous words". As an example of the second he offered the Greek letter "chi", so possibly the sound was similar to the "ch" in a Scottish "loch". (For the regular "ch" sound in "urchin" he used a combination of "t" and "sh", thus "urtshin".) Again we see the same pairing of voiced and

[1] In English, more often than in some other languages, the sounds associated with a character change according to circumstance: thus the "c" in "cat" does not sound the same as the "c" in "mice."

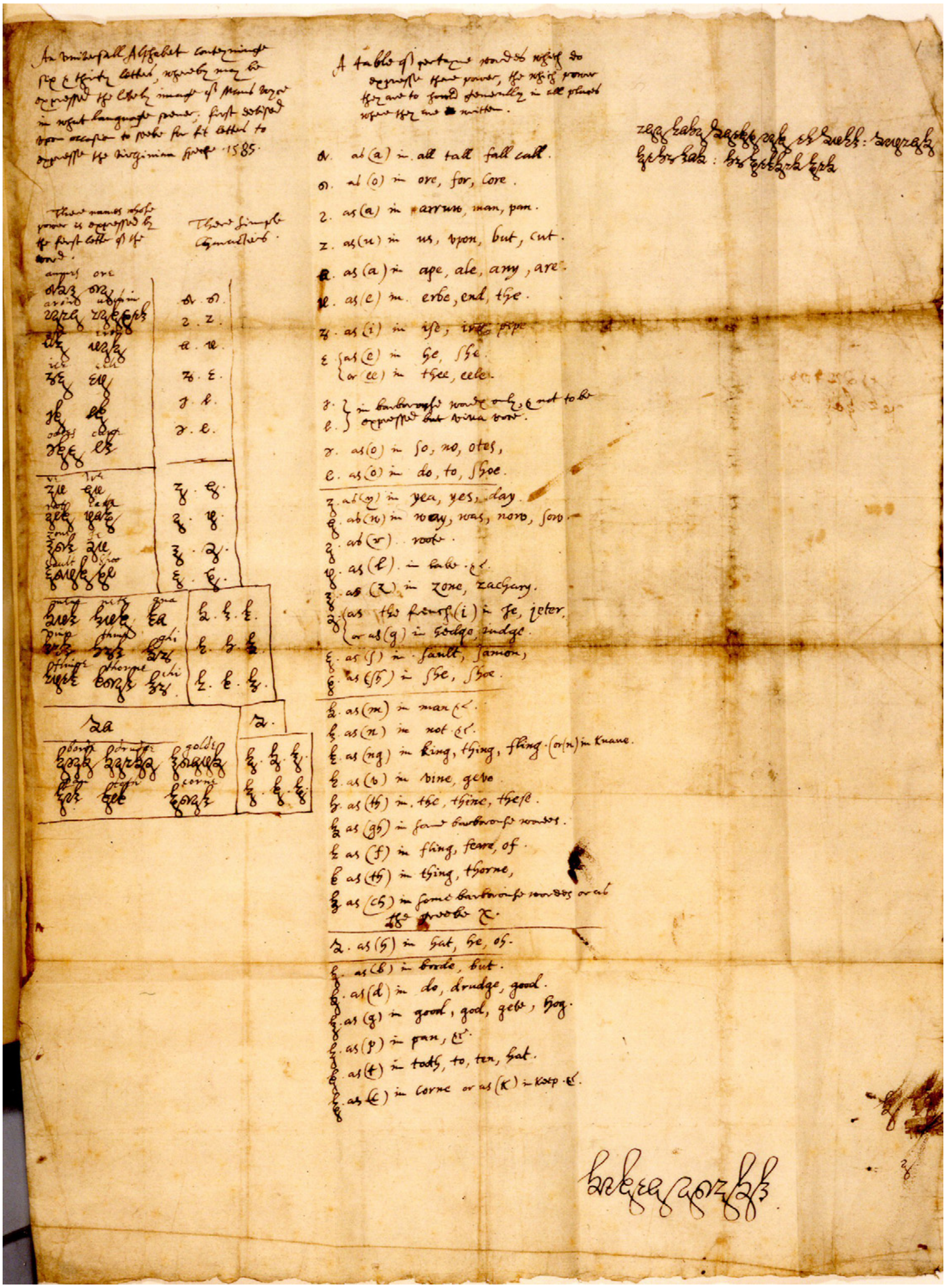

Fig. 1. Harriot's Algonquin alphabet, 1585, reproduced by kind permission of the Governing Body of Westminster School.

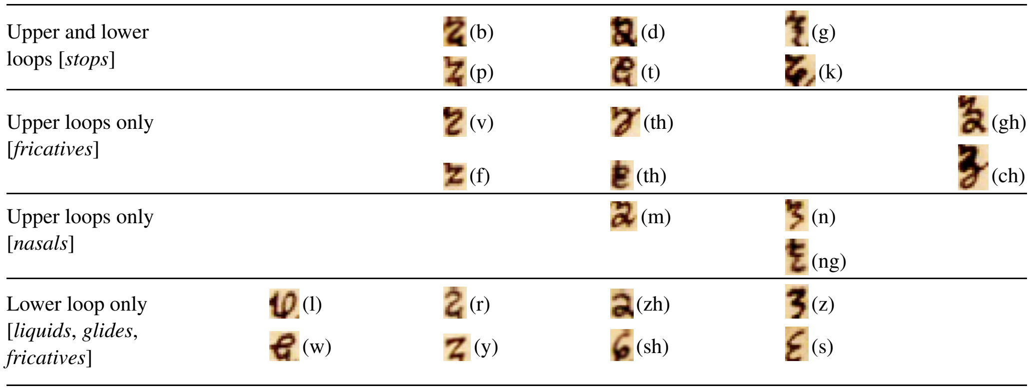

Fig. 2. The central elements of Harriot's symbols.

unvoiced sounds, and a similar pattern of furthest forward to furthest back as we move from left to right. The third row contains the *nasals*: "m", "n", "ng".

The remaining sounds, to which Harriot assigned only a lower loop, are those in which little or no contact is made anywhere in the mouth: the *liquids*, "l", "r"; the *glides*, "w", "y"; and the remaining fricatives, "zh" (Harriot's French "j"), "z", "sh", "s". Similar pairings of voiced and unvoiced sounds are seen here as in the previous rows. The final consonant, "h", has a symbol of its own (not shown).

Thus, although at first sight the letters look like random and unmemorable squiggles, it is clear that they are in fact very precisely constructed. Harriot's alphabet is a well-thought-out attempt to represent the sounds of human speech, using symbols and combinations of symbols to encode information about the position and formation of each sound in the mouth.

Salmon in 1992 recognized Harriot's use of upper and lower loops, and in the case of "t" and "d", and "k" and "g", also the partial reversing of the central elements, but he did not analyze the central elements of the other symbols and so missed some of the most important features of the system as a whole. Both he and Shirley correlated Harriot's symbols with their modern phonetic counterparts to expose the phonetic nature of his alphabet, but in doing so they to some extent obscured what Harriot himself was trying to do. Harriot's arrangement of the key sounds in pairs or triples was different from the modern classification, and by trying to force his alphabet into a modern mould, Shirley and Salmon missed both the "front to back" pattern in Harriot's symbols, and the combinatorial element that distinguishes them from those of any ordinary alphabet.

Like Shirley a few years earlier, Salmon suggested that Harriot borrowed some of his symbols from cossist algebra. More recently Michael Booth has based his own analysis of the alphabet on the same supposition [Booth, 2003]. The cossist symbols N, R, Z, C, however, are simply the first letters of the words for number (*numero*), a root or thing (*radix* or *res*), its square (*zensus*), and its cube (*cubus*). It is clear from the foregoing analysis that Harriot's symbols are completely different from these in construction and meaning, and that any resemblance in form is no more than coincidental.

Nevertheless, I would claim that Harriot's alphabet as a whole *is* algebraic, in a deeper and much more important sense than has previously been recognized. Algebraic symbolism must be capable of conveying two kinds of information: it must denote and distinguish the objects we are concerned with, and it must make visible some of the relationships between them. Harriot's system does both. Every sound has its own distinct symbol, and with only a few exceptions one can construct the correct symbol for a given sound from a simple set of rules. Further, related symbols indicate that the sounds themselves are related in a certain predefined way. What Harriot wrote out in 1585 was not just a phonetic alphabet but something very like a phonetic algebra.

We do not know whether Harriot was already studying mathematics in 1585, but his Algonquin alphabet relates to his later mathematics in three very important ways. First, it demonstrates Harriot's remarkable ability to express new ideas symbolically, something that is seen repeatedly in his mathematical writing. Second, his tabular arrangement of symbols on the page shows related pairs and triples, as well as his use of loops, without any further explanation. In a similar way, Harriot's mathematics is almost wordless, because he expects (and he is almost always right) that his reader will be able to *see* what he is doing either by following a symbolic argument, or from the layout of his material on the page. Communication between Harriot and his reader is much more often through visual imagery than through words. Third, we see in Harriot's alphabet his interest in combinations. From just three pairs of related signs, together with upper and/or lower loops, he could immediately create 18 different consonant symbols and, with only minor variations, all of them appear. To those he had to add further symbols for the awkward or "foreign" sounds that do not quite fit his system, but he managed to reduce almost to a minimum the number of shapes that have to be learned or remembered.

In its demonstration of symbolism, visual imagery, and combinatorial skill, I believe that Harriot's Algonquin alphabet of 1585 was a significant precursor to his later mathematical writing. The remainder of this paper will offer some examples to show how features first discernible in his alphabet were to reappear repeatedly in Harriot's mature mathematics.[2]

Symbolism in Harriot's mathematics

We do not know when Harriot first began to develop his mathematical symbolism. One of the earliest examples in his manuscripts is in connection with his experiments on falling bodies, which Shirley has dated to the 1590s [Shirley, 1983, 263–267]. Harriot assumed that different materials fall at different speeds, and to compare relative rates of fall of lead and wax, he supposed that he had the same weight of each. He took the volume of lead to be a, of wax $8a$, and the equivalent volume of air to be b. He also supposed that in a given time a certain weight of lead falls through a distance f and the same weight of wax through a distance g. He then wrote down this equation [BL Add MS 6788, f. 144v]:[3]

$$b - a \,.\, b - 8{,}a : f \,.\, g,$$

or, in modern notation,

$$(b - a) : (b - 8a) = f : g.$$

He immediately rearranged this to give[4]

$$bf - 8{,}fa = bg - ga$$
$$bf - bg = 8{,}fa - ga$$
$$8{,}f - g \,.\, f - g : b \,.\, a,$$

or, in modern notation again,

$$(8f - g) : (f - g) = b : a.$$

Thus in just a few lines Harriot had manipulated the equation so that he could use it to calculate $a : b$ or $f : g$ according to the data available. In this particular case he knew $f = 43\frac{25}{100}$ (feet) and $g = 42\frac{75}{100}$ (feet), and calculated $b : a = 10000 : 16\frac{1}{2}$, approximately. From earlier experiments he had found the density of lead to be 11351 (modern value on the same scale 11344) which enabled him to estimate the density of air as $18\frac{7}{10}$. His physics was wrong, but his algebra was correct.

[2] Michael Booth has argued that Harriot's encounter with the grammatical constructions of Algonquin also contributed to some of his mathematical innovations; see [Booth, 2002, 2003].

[3] This page is reproduced and transcribed in [Shirley, 1983, 265–267], but not accurately (for *by* read *bg*), and Shirley's interpretation of the calculations is also incorrect.

[4] Harriot's primary unknown was a, and so it was natural for him to write a as the final letter in any term that included it, just as we might now treat x. He did not worry about the ordering of b, f, g, which were all of subsidiary status.

A proposition in arithmetic

Harriot studied triangular numbers from the *Arithmeticorum libri duo* [1575] of Francisco Maurolico and the *Opus novum de proportionibus* [1570] of Girolamo Cardano, and his manuscripts contain just a few notes on both. Of particular interest here is a page from his manuscripts referring to two propositions from Maurolico [BL Add MS 6787, f. 246]. The first of these is Proposition 54, which in the original reads as follows [Maurolico, 1575, 24].[5]

> *Omnis triangulus octuplicatus cum unitate, conficit sequentis imparis quadratum.*
> Eight times any triangular number, plus one, makes the square of the next odd number.

As an example Maurolico gave the fifth triangular number, 15. It is easy to calculate that $8 \times 15 + 1 = 121 = 11^2$, the square of the sixth odd number. Harriot wrote the same proposition as follows. His notation $\left.\begin{array}{r} n \\ n+1 \\ \hline 2 \end{array}\right|$ is to be read as $\frac{n(n+1)}{2}$.

> *Sit latus trianguli n.*
> Let the side of the triangle be n.
> *Tum triangulus erit* $\left.\begin{array}{r} n \\ n+1 \\ \hline 2 \end{array}\right|$
> Then the triangle will be $\left.\begin{array}{r} n \\ n+1 \\ \hline 2 \end{array}\right|$
> *Unde propositionis demonstratio in notis logisticis ita se habet:*
> Whence the demonstration of the proposition is to be had in symbols thus:

$$\begin{array}{ccc}
\left.\begin{array}{r} 8n \\ n+1 \\ \hline 2 \end{array}\right| \begin{array}{r} +1 \\ 1 \\ \end{array} & = & \begin{array}{c} 2n+1 \\ 2n+1 \end{array} \\
\parallel & & \parallel \\
\left.\begin{array}{r} 8n + 8n \\ n \quad\;\; 1 \\ \hline 2 \end{array}\right| \begin{array}{r} +1 \\ +1 \\ \end{array} & & \left.\begin{array}{r} 4n \\ n \end{array}\right| \left.\begin{array}{r} +2n \\ 1 \\ +2n \\ 1 \end{array}\right| \left.\begin{array}{r} \\ \\ +1 \\ 1 \end{array}\right| \\
\parallel & & \parallel \\
\begin{array}{rrr} 4n & +4n & +1 \\ n & 1 & +1 \end{array} & & \left.\begin{array}{r} 4n \\ n \end{array}\right| \left.\begin{array}{r} +4n \\ 1 \end{array}\right| \begin{array}{r} +1 \\ +1 \end{array}
\end{array}$$

> *Ergo*
> Therefore

$$\left.\begin{array}{r} 4n \\ n \end{array}\right| \left.\begin{array}{r} +4n \\ 1 \end{array}\right| \left.\begin{array}{r} +1 \\ 1 \end{array}\right| = \left.\begin{array}{r} 4n \\ n \end{array}\right| \left.\begin{array}{r} +4n \\ 1 \end{array}\right| \begin{array}{r} +1 \\ 1 \end{array}$$

The left-hand column gives the working for eight times the nth triangular number, plus one, and the right-hand column gives the square of the $(n+1)$th odd number, and the totals at the bottom show that the two are equal, whatever the value of n. Of course, Harriot should not have written equality into the top row until he had arrived at equality in the bottom row, and perhaps he did not; it is impossible to tell from the finished page. In any case it is easy to find fault with such details when one is used to performing this kind of proof. When Harriot wrote this page he was breaking

[5] Harriot noted that the same problem was to be found in Stevin's edition of Diophantus and Viète's *Responsorum*; see [Stevin, 1585, 634; Viète, 1646, 371].

new ground, replacing Maurolico's specific examples with a completely general proof that, with only a little tidying up, is completely convincing.

Propositions from geometry

Around 1600 Harriot made an intensive study of the work of the French mathematician François Viète, and from then on, if not earlier, he frequently applied algebraic symbolism to propositions in geometry. The next two examples demonstrate this, one based on classical geometry, the other on a problem from Viète.

The first is Harriot's rewriting of the second book of Euclid in algebraic symbolism. On four sheets, Harriot offered all 14 propositions of Euclid II [BL Add MS 6785, ff. 153–156]. For the first 10 there are no diagrams at all. Propositions 1 and 2, for example, are written like this:

$$\text{pr[oposition]}\ \ 1)\quad \left.\begin{array}{r} b+c \\ \underline{\quad d} \end{array}\right| = bc+cd \qquad \left.\begin{array}{r} b+c+d \\ \underline{\qquad f} \end{array}\right| = bf+cf+df$$

$$2)\quad \left.\begin{array}{r} b+c \\ \underline{b+c} \end{array}\right| = \left.\begin{array}{r} b+c \\ \underline{\quad b} \end{array}\right| + \left.\begin{array}{r} b+c \\ \underline{\quad c} \end{array}\right|$$

William Oughtred in his *Clavis mathematicae* in 1631 also wrote these 14 propositions algebraically [Oughtred, 1631, 49–59]. Oughtred regarded this as the culmination of his teaching on analytic geometry, but his formulations are not nearly so elegant or concise as those of Harriot 30 years earlier.

The second example of Harriot's use of algebra in geometric problems is his reworking of a problem from Viète's *Effectionum geometricarum*, published in 1593. Viète's Proposition XII reads as follows [Viète, 1646, 234].

> *Sit data FD media trium proportionalium, data quoque GF differentia extremarum. Oportet invenire extremas.*
> Let there be given FD, the mean of three proportionals, and also GF the difference between the extremes. It is required to find the extremes.

Viète presented the diagram shown in Fig. 3 and explained how to construct the diameter BC from the given information. The line segments BF, DF, CF then represent the three required proportionals (by Euclid VI.13).

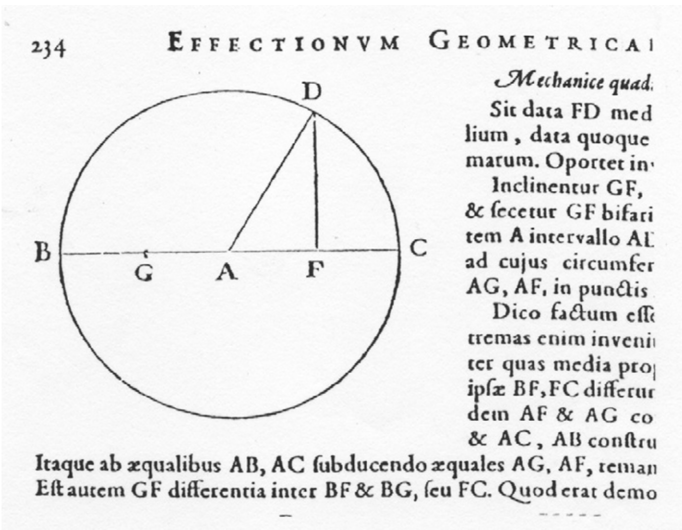

Fig. 3. Viète, *Effectionum geometricarum*, Proposition XII.

Harriot's version of this was as follows [BL Add MS 6785, f. 94]. As elsewhere in his manuscripts he used the notation a', d'', b''' to indicate that the three quantities a, d, and b are in geometric proportion. He also, as he explained, used the symbol $=\!=$ to indicate that b may be either added to or subtracted from a.

Data media trium proportionalium et differentia extremarum invenire extremas.
Given the mean of three proportionals and the difference between the extremes find the extremes.

$$\overbrace{a' \quad d'' \quad a \mathrel{=\!=} b'''}^{b} \tag{A}$$

Sit media data d et differentia b
Let the given mean be d and the difference b.

Et ponitur unus terminus ignotus a
And let one of the unknown terms be a.

Ergo alter erit $a \mathrel{=\!=} b$ hoc est
vel $a - b$ vel $a + b$.
Therefore the other is $a \pm b$, that is
either $a - b$ or $a + b$.

Ergo. Resolutio $aa \mathrel{=\!=} ba = dd$.
Therefore. The resolution $aa \pm ba = dd$.

Ergo per mechanicem $a = \sqrt{\frac{bb+4dd}{4}} \mathrel{=\!=} \frac{b}{2}$.
Therefore by construction.

Hoc est
That is

$$\begin{cases} \text{Minor} \quad a = \sqrt{\dfrac{bb+4dd}{4}} - \dfrac{b}{2} \\ \text{Maior} \quad a = \sqrt{\dfrac{bb+4dd}{4}} + \dfrac{b}{2} \end{cases}$$

As in his proof of the proposition from Maurolico, Harriot used words in this demonstration as sparingly as possible, mainly to define his notation. His initial diagrammatic statement (A) in ratio notation neatly encapsulated all the information in the question, and led directly to the defining equation, the "resolutio". Viète's construction gave him the solutions, the minor and major segments CF and BF, and Harriot noted that these correspond to the two possible solutions that arise algebraically.

Another construction method for such an equation is "completing the square", which Harriot also wrote out algebraically. Here, for convenience, he put $b = 2c$. From the relationship

$$\overset{\prime}{a} \quad \overset{\prime\prime}{d} \quad \overset{\prime\prime\prime}{a+2c}$$

he derived, as before, the equation

$$aa + 2{,}ca = dd.$$

He then continued as follows. (His notation $\sqrt[2]{}$, $\sqrt[3]{}$ below does *not* indicate square and cube roots, but roots of two or three combined terms.)

Mechanice secundum Diophantum et Mahometen.
Construction according to Diophantus and Mahomet.[6]

Adde utrinque parte aequationis cc.
Add to each side of the equation cc.

Ergo: $aa + 2{,}ca + cc = cc + dd$.
Therefore: $aa + 2ca + cc = cc + dd$.

[6] Mahomet was the name commonly used in the sixteenth century for al-Khwārizmī, being by that time almost all that was remembered of him.

Ergo: $\sqrt[3]{}/aa + 2, ca + cc = \sqrt[2]{}/cc + dd.$
Therefore: $\sqrt{aa + 2ca + cc} = \sqrt{cc + dd}.$

Hoc est: $a + c = \sqrt[2]{}/cc + dd.$
That is: $a + c = \sqrt{cc + dd}.$

Et per Antithesin $a = \sqrt[2]{}/cc + dd - c.$
And by Antithesis[7] $a = \sqrt{cc + dd} - c.$

Thus Harriot demonstrated that the ratio construction given by Viète and the method of completing the square both give rise to correct solutions of quadratic equations of the type $aa + 2ca = dd$. The manuscripts contain many other examples of Harriot exploring the relationships between geometry and algebra in this way, using notation of the kind shown here.

General formulae

In the examples above we saw Harriot using symbols to rewrite existing theorems, from Euclid, Maurolico, and Viète, and one could give many further instances, especially from his notes on Viète.[8] He also, however, used algebraic symbolism in mathematical investigations of his own, and here I offer two examples.

Crucial to the development of integration in the 17th century were the formulae for sums of squares, cubes, and higher powers of integers. Harriot's discovery of such formulae appears to have been inspired by his investigations into general triangular numbers. The pages where the formulae appear [BL Add MS 6782, ff. 239–240] carry several references to propositions from Maurolico, in particular to his Proposition 58, which states that the square of, say, the fifth triangular number is equal to the sum of cubes of the first five integers [Maurolico, 1575, 25]. Harriot's method of deriving the remaining formulae is not clear from the rough working on these two sheets but was probably similar to that later used by Fermat, who is also thought to have derived the rules for sums of powers from his knowledge of polygonal numbers [Mahoney, 1994, 229–233].

The page on which Harriot listed his formulae is headed "Ad aggregatum ʒ, *c*, ʒʒ, &c". ("Sums of squares, cubes, squares of squares, etc".), and they are written in a mixture of cossist and modern notation. For units, which are dimensionless, Harriot used the symbol 0; for lines, or "sides", he used a short straight line |; for squares, a small square □; for cubes, a small cube represented here by ⧉; and for fourth powers, the cossic notation ʒʒ. His formulae for the sum of the first *a* units, sides, squares, and so on, therefore appear as follows, where "*s*," stands for "sum of" and the integer to the right of each symbol is a multiplier. The formulae for sums of powers are as follows:

$$\begin{aligned}
s, 0; 1 &= 1, a \\
s, |; 2 &= 1, aa + 1, a \\
s, \square; 6 &= 2, aaa + 3, aa + 1, a \\
s, ⧉; 24 &= 6, aaaa + 12, aaa + 6, aa + 0, a \\
s, ʒʒ; 120 &= 24, aa, aaa + 60, aaaa + 40, aaa + 0, aa - 4, a
\end{aligned}$$

A second example of Harriot's skill in deriving general formulae comes from some work on arithmetic progressions, where he introduced the following notation:

p	*primus numerus* [first term]
u	*ultimus* [last term]
d	*differentia* [difference]
n	*numerus totorum* [number of all the terms]
s	*summa* [sum of terms].

[7] *Antithesis* was Viète's word for the process of adding or subtracting a term on each side of an equation.
[8] See also, for example, [Stedall, 2002, 92–94; Stedall, 2003, 44–123, 292–294].

Table 1
Harriot's formulae for the parameters of an arithmetic progression

cas [case]	*data* [given]	*quaesita* [sought]		
1.	$p\,u\,n$	d	$1'.n'' : d'''.u+d-p'''' \mid nd-1d = u1-p1 \mid d = \frac{u1-p1}{n-1}$	(1
		s	$1'.n'' : u+p.2,s \mid nu+np = 2,s1 \quad s1 = \frac{nu+np}{2}$	(2
2.	$p\,u\,d$	n	$------\ n = \frac{u1-p1}{d} + 1 \mid vel[\text{or}] \mid n = \frac{u1+d1-p1}{d}$	(1
		s	$d'.u+d-p'' : u+p'''.2,s'''' \mid [\ldots] \mid s = \frac{uu-pp+ud+pd}{2,d}$	(3
3.	$p\,u\,s$	n	$n = \frac{2,s1}{u+p}$	(2
		d	$d = \frac{uu-pp}{2,s-u-p}$	(3
4.	$p\,n\,d$	u	$u1 = nd+p1-1d$	(1
		s	$2,s11 = nnd+2np1-nd1$	(4
5.	$p\,n\,s$	u	$u = \frac{2,s1-np}{n}$	(2
		d	$d = \frac{2,s11-2,np1}{nn-1n}$	(4
...				

Harriot observed that given any three of these five quantities the other two could be found, and in a single page he gave all the necessary formulae for doing so (see Table 1) [BL Add MS 6782, f. 298]. For the first four cases he gave full derivations, but squeezed the working for each formula into a single line, which at first sight makes his table a little hard to read. To see how it is constructed, we may expand the first line as follows. On the left we have

Case 1. Given p, u, n, find d.

Harriot now used the fact that n times the difference d, added to the first term p, will give $u+d$, and it was natural for him as an early 17th-century mathematician to express this relationship using ratios. In modern notation we can write $1:n = d:(u+d-p)$; in Harriot's notation this was $1'.n'' : d'''.u+d-p''''$, which appears as the first entry in the line. The rest of the working may be written as below. Harriot included all but the second step in the same first line.

$$1:n = d:(u+d-p)$$
$$nd = u1+d1-p1$$
$$nd-1d = u1-p1$$
$$d = \frac{u1-p1}{n-1}.$$

The numeral 1 written after a letter, as in $u1$ or $p1$ and later $s11$, does not change the value, but Harriot kept it to maintain the homogeneity of each equation. In a note at the foot of the page he explained that 1 is to be read as a unit (*unitas*), and 11 and 111 as the square and cube of a unit, respectively.

After the first four lines, the formulae need no longer be derived from first principles but can be obtained from those already found. The numbers in the right-hand column indicate related formulae and show that Harriot had no difficulty in rearranging equations into whatever form he required.

After the opening shown in Table 1, Harriot's table continues with the remaining combinations $p\,d\,s$, $u\,n\,d$, $u\,n\,s$, $u\,d\,s$, $n\,d\,s$, and it is clear that he worked in a regular way through all combinations of three letters chosen from p, u, n, d, s, taking them in that order. This example therefore illustrates not only Harriot's skill in algebraic manipulation, but also his liking for systematic listings of combinations. We shall return to his interest in combinations shortly.

Investigations in symbols

It is clear from the fact that Harriot rewrote so much existing mathematics in his own notation that he found a symbolic mode of expression clearer and more helpful than the largely verbal presentations of his predecessors. But

Harriot went further: symbolism became for him not just a more concise way of writing, a kind of mathematical shorthand, but also an investigative tool. We have seen this to some extent already in his handling of formulae for arithmetic progressions, which he was able to manipulate into new forms without any reference to their meaning. Two further examples will serve to show him using symbols to open up some old mathematics in an entirely new way.

The first example is the Rule of False, an ancient method of problem-solving found in Indian, Chinese, and Arabic texts and known in western Europe from the thirteenth century onwards. It was taught by rote in many sixteenth-century textbooks of arithmetic. The method consists of making two (false) guesses at the solution, and then calculating the correct answer from the information so obtained. It is best demonstrated by an example, and Harriot chose the following problem from Robert Recorde's *The whetstone of witte*:[9] Alexander is two years older than Ephestio; their combined ages plus another four years give the age of Ephestio's father; the combined ages of all three give the age of Alexander's father (Calistemis) who is 96. Harriot denoted the age of Ephestio by a and wrote down the following information (see Fig. 4) [BL Add MS 6783, f. 316].

$$\begin{array}{cccc} \text{Ephestio} & \text{Alexander} & \text{Clyts} & \text{Calistemis} \\ a & a+2 & a+a+2+4 = & 96 \end{array}$$

It does not take long to work out by easy mental arithmetic or from a simple equation that $a = 22$. Harriot was not interested in the answer, however, but in the working of the Rule of False, and tested it by supposing the age of Ephestio first to be 8, and then 16. The first hypothesis gave him a total of 40, the second a total of 72, both short of the required 96. The Rule of False, applied by rote, now told him that the true answer should be

$$\frac{16 \times (96-40) - 8 \times (96-72)}{72-40} = \frac{16 \times 56 - 8 \times 24}{32} = \frac{896-192}{32} = \frac{704}{32} = 22.$$

Harriot set out this information as follows.

$$\begin{array}{c|lllllll} b & 8 & 8+2 & 8+8+2+4 & =40 & & & \\ & & & & & 56 & 896 & \\ a & a & a+2 & a+a+2+4 & =96 & 32 & & \frac{704}{32} = 22 = a \\ & & & & & 24 & 192 & \\ c & 16 & 16+2 & 16+16+2+4 & =72 & & & \end{array} \tag{B}$$

Next moving entirely into symbols, Harriot denoted the correct answer by k and the two false answers by d and f and created the following difference tables. The "ladder" sign separating the two halves of this and other tables is a sideways adaptation of Harriot's equals sign, and may be read as "is equivalent to":

$$\begin{array}{ccc||ccc} & & b & d & & \\ & a-b & & & k-d & \\ c-b & & a & k & & f-d \\ & a-c & & & k-f & \\ & & c & f & & \end{array} \tag{C}$$

It is clear that d, k, and f are calculated from b, a, and c, respectively, by the same linear relationship. In fact $d = 4b + 8$, $f = 4c + 8$, and $k = 4a + 8$, but the Rule of False does not require these equations to be identified explicitly. Taking differences eliminates constant terms, and we are left with the observation that $f - d$ is in the same ratio to $c - b$ as, say, $k - d$ to $a - b$. Harriot was therefore able to write down

$$f \overset{\prime}{-} d, k \overset{\prime\prime}{-} d : c \overset{\prime\prime\prime}{-} b, a \overset{\prime\prime\prime\prime}{-} b$$

[9] *The whetstone of witte* [1557] is not paginated. The problem chosen by Harriot is in the section headed "The arte of Cossicke numbers", where Recorde solved it by means of a simple linear equation.

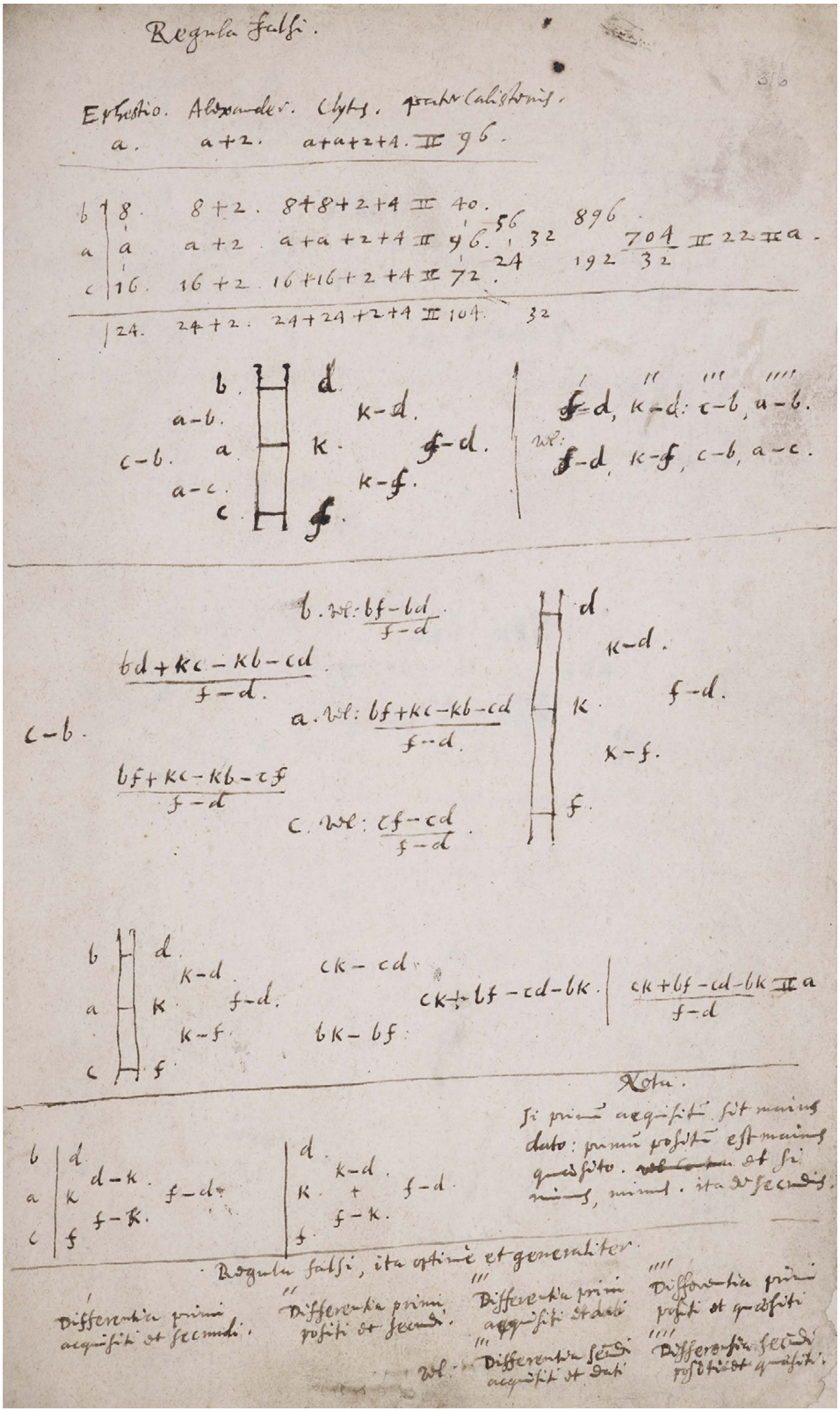

Fig. 4. The Rule of False, from BL Add MS 6783, f. 316, reproduced by kind permission of the British Library.

and

$$f \overset{\prime}{-} d, k \overset{\prime\prime}{-} f : c \overset{\prime\prime\prime}{-} b, a \overset{\prime\prime\prime\prime}{-} c$$

which in modern notation are

$$(f-d):(k-d)=(c-b):(a-b)$$

and

$$(f-d):(k-f)=(c-b):(a-c).$$

These equations allowed him, without too much difficulty, to calculate $a-b$, $a-c$, and a itself in terms of b, c, d, f, and k in the difference table on the left-hand side of (C):

$$\begin{array}{ccc|ccc} & & b \text{ or } \dfrac{bf-bd}{f-d} & d & & \\ & \dfrac{bd+kc-kb-cd}{f-d} & & & k-d & \\ c-b & & a \text{ or } \dfrac{bf+kc-kb-cd}{f-d} & k & & f-d \\ & \dfrac{bf+kc-kb-cf}{f-d} & & & k-f & \\ & & c \text{ or } \dfrac{cf-cd}{f-d} & f & & \end{array} \tag{D}$$

Now Harriot carried out once again the procedure of the Rule of False as in (B), but this time entirely algebraically:

$$\begin{array}{c|c|ccccc} b & d & & & & & \\ & & k-d & & ck-cd & & \\ a & k & & f-d & & ck+bf-cd-bk & \left| \; \dfrac{ck+bf-cd-bk}{f-d} = a \right. \\ & & k-f & & bk-bf & & \\ c & f & & & & & \end{array} \tag{E}$$

It is clear that the answer he reached in this last calculation, namely

$$\frac{ck+bf-cd-bk}{f-d},$$

is the same as the value of a displayed in (D), which confirms that the Rule of False does indeed give the correct result.

This is not, as might at first appear, a longwinded solution to an easy problem, but an investigation into how the Rule of False actually works, and Harriot's symbolism and layout render the structure of the problem and the process of solving it completely transparent. The only words on the page are some brief notes at the end about the changes needed if $k-d$, for example, turns out to be negative.[10]

The final example is again based on mathematics that was already well known, the calculation of compound interest. Over four related pages, which are now bound together, though not necessarily in the order in which they were written, Harriot investigated the behavior of the interest on an initial sum, when it is added at more and more frequent intervals [BL Add MS 6782, ff. 67–70]. He began with an easy example, the interest on £100 at a rate of 10% per

[10] For further examples of the way Harriot used page layout to provide a visual image of his thinking, see his analysis of polynomial equations in [Stedall, 2003, 124–287].

annum. If interest is added at the end of the year the sum becomes £$100(1+\frac{1}{10})$ or £$(100+10)$. If it is added every six months, however, the sum will be £$100(1+\frac{1}{20})^2$ or £$(100+10+\frac{1}{4})$; if three times in the year £$100(1+\frac{1}{30})^3$ or £$(100+10+\frac{1}{3}+\frac{1}{270})$; and so on. Harriot made these calculations for interest added up to five times a year, but then, as so often, he moved into symbols. If interest is added n times a year, then the sum will be, in modern notation, £$100(1+\frac{1}{10n})^n$ or

$$£\left(100+\frac{10n}{n}+\frac{n(n-1)}{2n^2}+\frac{n(n-1)(n-2)}{6.10n^3}+\frac{n(n-1)(n-2)(n-3)}{24.10^2n^4}+\cdots\right).$$

Harriot wrote this as shown below [BL Add MS 6782, f. 69]. In each column the terms above the line are to be multiplied to give the numerator, and those below the line are multiplied to give the denominator, a method of vertical stacking that Harriot often used to indicate multiplication of several terms:[11]

$$100 + \begin{array}[t]{c} 10 \\ n \\ \hline 1,n \end{array} + \begin{array}[t]{c} 1,n-1 \\ 1,n \\ \hline 2,n \\ n \end{array} + \begin{array}[t]{c} 1,n-2 \\ 1,n-1 \\ 1,n \\ \hline 6,n \\ n \\ n \\ 10 \end{array} + \begin{array}[t]{c} n-3 \\ n-2 \\ n-1 \\ n \\ \hline 24,n \\ n \\ n \\ n \\ 10 \\ 10 \end{array} + \begin{array}[t]{c} n-4 \\ n-3 \\ n-2 \\ n-1 \\ n \\ \hline 120,n \\ n \\ n \\ n \\ n \\ 10 \\ 10 \\ 10 \end{array} \ \&c.$$

Harriot now did something quite remarkable. He assumed that as interest was added more and more frequently he could treat the process as continuous rather than discrete, so that he could calculate "The summe of interest upon interest continually for every instant the whole year". He also assumed that as n became very large the terms involving n in the numerators and denominators would cancel each other out. Thus the sum becomes (as Harriot wrote it)

$$100+\frac{10}{1}+\frac{1}{2}+\frac{1}{6,0}+\frac{1}{24,00}+\frac{1}{120,000}+\frac{1}{720,000}+\frac{1}{5040,00000}.$$

Returning to pounds, shillings, and pence, the second and third terms amount to £10 and 10 shillings.[12] The fourth and fifth add a further $4\frac{1}{10}$ pence, and Harriot observed that all the remaining terms together would not add as much as another $\frac{1}{10}$ of a penny, so that £10 plus 10 shillings and $4\frac{2}{10}$ pence is an upper limit for the interest added.

In modern parlance Harriot has allowed n to "tend to infinity" and has found that the limit of

$$\left(1+\frac{1}{10n}\right)^n$$

is the infinite series that we now recognize as $e^{1/10}$:

$$1+\frac{1}{10}+\frac{1}{2!10^2}+\frac{1}{3!10^3}+\frac{1}{4!10^4}+\cdots.$$

[11] This is not the only occurrence in Harriot's manuscripts of the binomial coefficients expressed algebraically. They also appear at the beginning of a treatise entitled "De numeris triangularibus", written down in 1618 but based on work done much earlier. There Harriot deduced the general form by inspection from what is now known as "Pascal's triangle" [BL Add MS 6782, f. 110].

[12] In Harriot's day, as in British currency until 1971, there were 240 pence, or 20 shillings, to a pound.

This is a very interesting calculation in its own right, but it also anticipates exactly the procedure that Brook Taylor used in his *Methodus incrementorum* more than a century later in his derivation of Taylor series [Taylor, 1715, 21–23]. Taylor described it a little differently—instead of calculating "interest upon interest" he worked with increments of increments—but the underlying mathematics and the formulae that arise are identical. According to Taylor the coefficients

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

were given by the "Newtonian theorem", that is, from Newton's formulae for general binomial coefficients. By that time neither he nor anyone else was aware that Harriot had long ago found such formulae for the case where n is an integer, and by letting n become infinitely large had also discovered a special case of the exponential series.

In the four sheets of Harriot's that survive on this subject we can see that he experimented a little further. On one of them, for example, he calculated "Interest upon interest for 7 yeares" on a sum of £100 at interest of 10% [BL Add MS 6782, f. 67, partly reproduced in Pepper, 1968, 410]. Here, in his initial calculations, he replaced the principal sum of 100 by the general term bb and the rate of interest by b. Once again, he allowed n to become indefinitely large so that he could find "The summe of interest upon interest continually for every instant", which amounts in this case to £201 plus 7 shillings and $6\frac{6}{100}$ pence. A more general table, in which the principal sum is b and the rate of interest $\frac{1}{c}$, appears on a third sheet, though it was not taken to any conclusion, or at least not there [BL Add MS 6782, f. 70].

It is clear that Harriot had a procedure that could be applied to interest calculations generally and that he was capable of writing his argument entirely algebraically. Indeed, he could hardly have found the limiting case, of interest taken "continually for every instant", *without* writing his fractions symbolically. His initial numerical calculations, for example, led him to the sequence of fractions $0, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \dots$ (in Harriot's calculations they appear as $\frac{10}{40}, \frac{30}{90}, \frac{60}{160}, \frac{10}{25}$), and it is not at all obvious, without taking a good many terms, that this sequence converges to any limit at all. Once the general term is written as $\frac{(n-1)n}{2nn}$, however, it is plain that for large values of n the fractions must be just below but very close to $\frac{1}{2}$. Here, as also in his investigation of the Rule of False, Harriot's symbolism gave him far more than just an efficient way of writing out the calculation. It also gave him insights into the underlying mathematics and enabled him to pursue the problem much further than numerical calculation alone would have allowed.

Combinations

We have already seen Harriot's systematic use of combinations in his Algonquin alphabet, and later also in his list of formulae for arithmetic progressions. A further example, of a quite different kind, is a diagrammatic attempt to sketch all the possible relative positions of three circles: separated, touching, intersecting, or contained inside one another [BL Add MS 6785, f. 65]. This arose from Harriot's study of Viète's *Apollonius Gallus* (1600), and so had a serious mathematical purpose, but the sheet on which he drew the arrangements has the charming appearance of a page of floating bubbles. The bubbles themselves, and Harriot's efforts to count them, soon got out of hand, but inspection reveals that, at least to begin with, he approached the problem in an orderly way, by drawing first all possible relative positions of two circles. Then, for each pair, he systematically added a third circle, in such a way that the original circles touch it on the inside, then on the outside, then one inside and one outside, then one but not the other, and so on. Multiple variations and repetitions soon make the counting difficult but, once again, we see Harriot trying to run systematically through the possible options.

The main purpose of this section is to look at what Harriot had to say explicitly on combinations, most of which is now to be found in BL Add MS 6782. This includes nine sheets headed either "Of combinations" or "Of transpositions" [BL Add MS 6782, ff. 33–41] and related material from elsewhere in the same volume [BL Add MS 6782, ff. 30, 32–54, 180–185, 331–332].[13]

[13] Ian Maclean has studied Harriot's work on combinations in relation to the social, political, and religious context of 16th-century England. In his description of Harriot's mathematical work on combinations, however, he has conflated material from the sheets headed "Of combinations" with material from a separate treatise with a different purpose entitled "De numeris triangularibus et inde de progressionibus arithmeticis", with further material from random extraneous pages, and even with Harriot's speculations on the infinite [Maclean, 2005, 76–77]. The result is a somewhat confusing account of Harriot's mathematical achievements in these separate areas.

First, we will consider a single sheet with the Latin heading “De combinationibus” [BL Add MS 6782, f. 30], which in the present ordering of the manuscripts comes shortly before the nine mentioned above. In it, Harriot demonstrated his familiarity with the properties of what are now called binomial coefficients. At the left of the page are two versions of “Pascal’s triangle”, one set out in the form of an upright isosceles triangle, the other as a right-angled triangle, and both taken as far as the row 1, 5, 10, 10, 5, 1. On the right of the page, powers of $(b+c)$ up to $(b+c)^4$ are calculated by long multiplication, and Harriot has marked with an asterisk the lines in which the coefficients 1, 2, 1 or 1, 3, 3, 1 or 1, 4, 6, 4, 1 appear. Similar calculations for $(1+1)^4$ yield the coefficients alone, uncluttered by powers of b or c:

$$
\begin{array}{rrrrr}
 & & & 1 & +\,1 \\
 & & & 1 & +\,1 \\
\hline
 & & & 1 & +\,1 \\
 & & 1 & +\,1 & \\
\hline
 & & 1 & +\,2 & +\,1 \\
 & & & 1 & +\,1 \\
\hline
 & & 1 & +\,2 & +\,1 \\
 & 1 & +\,2 & +\,1 & \\
\hline
 & 1 & +\,3 & +\,3 & +\,1 \\
 & & & 1 & +\,1 \\
\hline
 & 1 & +\,3 & +\,3 & +\,1 \\
1 & +\,3 & +\,3 & +\,1 & \\
\hline
1 & +\,4 & +\,6 & +\,4 & +\,1
\end{array}
$$

Finally, the following brief calculation demonstrates why the sums of the coefficients (each marked with $\star$) are always powers of 2:

$$
\begin{array}{cl}
1+1 & \\
1+1 & \\
\hline
2+2 & \star \\
1+1 & \\
\hline
2+2 & \\
2+2 & \\
\hline
4+4 & \star \\
1+1 & \\
\hline
4+4 & \\
4+4 & \\
\hline
8+8 & \star
\end{array}
$$

Elsewhere Harriot used long multiplication of binomials in a similar way to discover the combinations of 2, 3, or 4 letters, namely, p and m; or of p, m, and f; or of p, m, f, and s [BL Add MS 6782, ff. 181 and 48v].[14] The first two calculations are as follows:

$$
\left.\begin{array}{r} p+\cancel{p} \\ m+\cancel{m} \\ \hline \end{array}\right| = \begin{array}{ll} pm & +\cancel{p}m \\ & +p\cancel{m}+\cancel{p}\cancel{m} \end{array}
$$

[14] From a third sheet with similar calculations [BL Add MS 6786, f. 291], we know that the letters p, m, f, s, and there also a, stand for *pondus* (weight), *magnitudo* (size), *figura* (shape), *situs*, (position), and *altitudo* (height); but when or why Harriot was interested in the possible combinations of these attributes I do not know.

$$\left.\begin{array}{r} p+\not{p} \\ m+\not{m} \\ f+\not{f} \end{array}\right| \begin{array}{llll} = pmf & +\not{p}mf & & \\ & +p\not{m}f & +\not{p}\not{m}f & \\ & +pm\not{f} & +\not{p}m\not{f} & \\ & & +p\not{m}\not{f} & +\not{p}\not{m}\not{f} \end{array}$$

If we read Harriot's $p+\not{p}$ as "p or not-p" then his calculations show immediately how the various combinations of p, m, and f arise, and the columns show exactly how many contain three, two, one, or none of the letters. There are no words at all on the page, and none are needed.

We now turn to the pages headed "On combinations" [BL Add MS 6782, ff. 33–38]. Harriot began (folios 33 and 34) by listing all possible combinations (or "complications", as he sometimes called them) of one, two, three, four, five, six, or seven letters and did so very systematically. His list of four-letter combinations, for instance, is as follows:

$$\begin{array}{r|l} \underline{abcd} & \\ & \\ a & \\ b & \\ c & 4 \\ d & \\ \hline\hline ab & \\ ac & \\ ad & \\ \hline bc & 6 \\ bd & \\ \hline cd & \\ \hline\hline abc & \\ abd & \\ acd & 4 \\ bcd & \\ \hline\hline abcd & 1 \\ \hline\hline 15 & \end{array}$$

A small chart at the bottom of folio 33 shows also how to build up each column from the previous one and at the same time illustrates why the number of combinations, as Harriot was counting them here, doubles plus one for each additional letter.

$$\begin{array}{ll} ab & abc \\ \hline a & a \\ b & b \\ ab & ab \\ & a|c \\ & b|c \\ & ab|c \\ \hline & c \end{array}$$

Next (folio 35), Harriot again gave lists of combinations of one, two, three, four, and five letters, now arranged in this new fashion. At the bottom of the page he finally observed that in certain cases the null combination should also be counted, thus,

z is a, or not
z is a or b or ab or nothing
z is a or b or c or ab or ac or bc or abc or nothing
&c

and noted that the number of combinations now doubles for each additional letter. (Similar information is also contained in another and larger chart in the same volume, but in the present ordering of the manuscripts it is somewhat separated from "On combinations" [BL Add MS 6782, f. 180].)

The next two pages of "On combinations" (folios 36 and 37) contain lists of binomial coefficients for expansions up to $(1+1)^{24}$. This is followed (folio 38) by a statement of the "Generall rule" that the total number of "all the complications of any Number of Species" is a power of 2.

Harriot then turned (in folios 39 to 41) to what he called "transpositions" but what we would now call "permutations" or "arrangements". Harriot's meaning is very clear from his description (folio 39), which at the same time gives a neat inductive argument.

> For suppose the number of transpositions of 3 species [letters] that is $a\ b\ c$ to be 6. The next number to be transposed is of things let be $a\ b\ c\ d$. Now d, in respect of $a\ b\ c$ hath four places. That is he may be met after c: after b: after a: or before a. So many places it hath with acb and the rest of the 6. Therefore 4 times 6 thus 24 is the number of transpositions of 4 species.

This verbal description is preceded, however, by yet another of Harriot's visual aids, lists of "transpositions" of one, two, three, four, or five letters. Here is the list for three letters:

$$\begin{array}{c} abc \\ acb \\ cab \\ \hline bac \\ bca \\ cba \\ \hline\hline 6 \end{array}$$

The construction of the table anticipates the description that follows it. Starting with ab, Harriot has placed c after b, then after a, then before a. After that he could proceed no further, so he drew a line under this set of three arrangements and began again from the second possible arrangement of a and b, namely ba. Once he has used up both possible arrangements of a and b, he has produced all six possible arrangements of a, b, and c.

On the same page (folio 39) Harriot listed in a side table what he called "single variations":

$$\begin{array}{l} abcd \\ bcda \\ cdab \\ dabc \\ \hline abcde \\ bcdea \\ cdeab \\ deabc \\ eabcd \\ \hline \end{array}$$

These are just cyclic arrangements of the original letters, and Harriot noted that "There are so many as there are species". (On a separate sheet [BL Add MS 6782, f. 44v] Harriot drew a small circle with the letters a, b, c placed at equal intervals around it, and under it wrote abc, cab, bca.)

The next page (folio 40) is headed "Combinations and transpositions together", and here Harriot calculated the number of all possible combinations and permutations of up to seven letters. The first row of (F) shows the number of possibilities for one letter chosen from up to 7; the second row shows the number of possibilities for two letters chosen from up to 7; the third row shows the same information for three letters; and so on. There is no further explanation, though in the original table a number of diagonal lines have been added between the entries, indicating, for example, that 840 in the seventh column is 7 times 120 from the sixth column, which is in turn 6 times 20 from the fifth column, and so on:

1	2	3	4	5	6	7	
1	2	3	4	5	6	7	1.
	$\underline{2}$	6	12	20	30	42	2.
	4	$\underline{6}$	24	60	120	210	3.
		15	$\underline{24}$	12	360	840	4.
			64	$\underline{120}$	720	2520	5.
				325	$\underline{720}$	5040	6.
					1956	$\underline{5040}$	7.
						13699	

(F)

The final page of the set of nine (folio 41) appears to continue from the previous ones but may be a separate investigation. As so often in Harriot's papers, what began as a well-ordered treatise ends in a confusion of rough work and new ideas, so that it is impossible to tell where one train of thought ends and another begins. Folio 41 is headed "Of combinations and transpositions of the numbers UV XYZ", where the last five letters are symbols from his Algonquin alphabet. They are the symbols that represent the English sounds "i" (as in "ill"),"n", "d","i" (as in "ize"),"s", and so Harriot's heading reads "Of combinations and transpositions of the numbers in dice". Such appearances of his Algonquin script are rare, and why Harriot should have written these two words in code I do not know.

A large table with 6 columns and 36 rows, filling most of the page, shows all possible outcomes of throwing a single die three times in succession. Below it, there is a list of all possible sums, from 3 to 18, and the number of ways each one can arise; the sums 10 and 11 are marked as the most likely, with 27 possibilities for each. Similar lists appear nearby for the sums on a die thrown four or five times in succession (folios 41 and 40v). Further rough work and related calculations appear on several subsequent pages (folios 42–54). Elsewhere in the same volume (folios 184 and 185) there are completed tables for up to six throws, again showing the number of ways the various sums can be obtained. For six throws (for which there are $6^6 = 46,656$ possible outcomes), the totals are carefully broken down into subtotals for the cases in which two, three, four, five, or six numbers are repeated.

We know that Harriot read Cardano's *De proportionibus* and that he found there some of the characteristic properties of what we now call binomial coefficients, together with an explanation of their relevance to calculations of combinations [Cardano, 1570, Propositions 137, 170]. However, he also carried out many calculations of his own and worked out for himself the basic principles of a theory of combinations and permutations. One hundred years later, in his *Ars conjectandi*, Jacob Bernoulli devoted a chapter to calculating the possible combinations and permutation of the seven letters *aaaabbbcc*, allowing for repetitions [Bernoulli, 1713, 133–137]. He did so, he said, because he had not seen this problem treated by other authors, and of course he could not have known of Harriot's calculations, which by then were lying forgotten at Petworth House. Bernoulli's procedure was very similar to Harriot's, adding new letters to the end of existing combinations and then moving them leftward into each possible position. Harriot did not treat repetitions in "On combinations" but he certainly did so in his calculations of sums on dice. One can only speculate as to how much sooner a mathematical theory of probability might have emerged if Harriot's work had been published in the early 17th century.

Concluding remarks

A detailed analysis of how and why mathematical symbolism developed so rapidly at the beginning of the 17th century is beyond the scope of this paper, but my study of Harriot's manuscripts leads me to the following remarks, on the process of symbolization in general and Harriot's symbolism in particular.

First we must note that some basic symbolism was already in use in the 16th century: the cossist letters N, R, Z, C to represent numbers, roots, squares, and cubes; Bombelli's superscripts for powers [1572]; Viète's single letters for lengths of lines, either known or unknown. All of these had severe limitations, however. The use of specific letters or words for squares and cubes, in both cossist notation and Viète's, made it difficult to generalize to more than three dimensions. Bombelli's index notation, adopted also by Stevin [1585] and later Girard [1629], was more adaptable, but only so long as equations contained no more than a single variable. Harriot was the first to devise a notation in which any number of unknown quantities could appear to any power. During the 1620s his first editor, Nathaniel Torporley, already began to abbreviate Harriot's *aaaa*, for example, to a^{IV} [CUL 9597/17/28, f. 69], but unfortunately this convention was not followed by his second editor, Walter Warner, and did not appear in the printed edition of Harriot's work, the *Artis analyticae praxis* [1631].

Harriot worked many years earlier than the other most prolific early 17th-century inventors of symbolism: Oughtred, Hérigone, and Descartes. His notation was no less effective than theirs, but did not appear in print until 1631, and then in a text whose circulation was probably limited. Unlike Oughtred's *Clavis* [1631] or Hérigone's *Cursus* [1634–37], the *Praxis* was a record of advanced research, and not a textbook for beginners. The influence of Descartes' *La géométrie* [1637] was so great that much of its notation became standard, though the English symbols for equality and inequality survived. In England, Harriot's *a* was not pushed aside by Descartes' *x* until quite late in the 17th century. Kersey taught Harriot's notation, for example, including the use of repeated letters for powers, in his *Elements of algebra* published in 1673, and described it (in comparison to that of Descartes) as "plainest for learners" [Kersey, 1673, 4–5].

Florian Cajori, in his 800-page *A history of mathematical notations*, described Harriot's notation as found in the *Praxis*, but not that of the manuscripts, which he had not seen [Cajori, 1928–29, i, 199–201]. In the example he quoted [from Harriot, 1631, 65], Cajori misunderstood Harriot's equals signs, which are in fact similar to the "ladder" symbols in Fig. 4, (C), (D) and (E) above, but he did remark on Harriot's effective use of page layout. Fifty years later, Johannes Lohne, the most thorough 20th-century analyst of Harriot's manuscripts, claimed that in the invention of notation Harriot excelled all others except Leibniz [Lohne, 1979, 295]. He also suggested that if Harriot's work had become known sooner, subsequent mathematical textbooks might have looked very different. This must remain a matter of speculation, but of Harriot's priority and inventiveness there is no longer any doubt.

The question of *why* so much mathematical notation emerged so rapidly in the early 17th century is not easy to answer. To those accustomed to mathematical symbolism, its advantages are obvious, and in 1647, William Oughtred, another English mathematician who enjoyed inventing new notation, wrote an elegant accolade to its value [Oughtred, 1647, Preface]:

> For this specious and symbolicall manner, neither racketh the memory with multiplicity of words, nor chargeth the phantasie with comparing and laying things together; but plainly presenteth to the eye the whole course and process of every operation and argumentation.

Harriot would have agreed, but Oughtred's words were actually written for those who complained that they found this new style of mathematics difficult. In wondering why symbolic expression was not established earlier, we should not underestimate the sophistication of mathematical understanding that could be achieved without it: Cardano's *Ars magna* in 1545 and Viète's treatises of the 1590s were all profoundly influential, yet their expositions were primarily, sometimes entirely, verbal.

One reason, I suggest, for the development of new notation was that those who grappled with such texts found that they needed to rewrite the contents for themselves in what seemed to them a clearer form. Thus Bombelli's notation was his response to Cardano [1545] and Harriot's was, repeatedly, a response to Viète. Further support for this argument is that many other readers and admirers of Viète can similarly be seen transcribing his ideas into symbols: Oughtred [1631], Hérigone [1634–37], and Wallis (in manuscript sheets inserted into his copy of Viète's *Opera mathematica* of 1646) [see Stedall, 2002, 81. The major exception was Fermat, who was also profoundly influenced by Viète, but to the end of his life resisted the move toward symbolism.[15]

A second reason for the emergence of symbolic notation from 1600 onwards was the new understanding, again arising largely from the work of Viète, of the interchangeability of geometry and algebra. Once one sees that propositions in geometry can be expressed as equations, it becomes natural, indeed almost essential, to represent lines, areas, or volumes, by symbols. Viète and even Fermat began to do so, and most of the other early expositors of analytic geometry, Harriot, Oughtred, and Descartes, took the process much further.

[15] As late as 1658 Fermat advised Wallis to banish symbols, or species, in analysis and to return to traditional Euclidean and Apollonian constructions. (*Monemus ... ut sepositis tantisper speciebus Analyseos problemata Geometrica via Euclideana et Apolloniana exequantur* [Wallis, 1658, 191; or Wallis, 2003, 496].)

Harriot's development of mathematical notation was thus part of a movement that was to gather pace rapidly over the next half century. At the same time Harriot stands apart in being several years ahead of any but Viète, and in the lengths to which he was prepared to go. Among early 17th-century writers no one but he was willing to dispense with words so completely, and he had no counterpart in this until John Pell (a great admirer of Harriot) also developed wordless mathematics some 40 years later [Malcolm and Stedall, 2005, 267–270]. The details of which particular bits of Harriot's notation survived and which did not are less important than the fact that as early as 1600 he was already writing purely symbolic mathematics. With hindsight this can be seen as an early manifestation of a more general trend, but at the same time it can be recognized that it sprang from Harriot's innate interest and ability in such matters, the evidence for which begins with his Algonquin alphabet of 1585.

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